Estimate Differential/Common-Mode Properties from RLGC Matrices with Minor Asymmetry

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1 Relations Based on the Transmission-line Theory

Problem Statement: In some cases, the RLGC matrices of a 2-line transmission-line system contain minor asymmetry due to numerical approximations. It is desirable to estimate the differential impedance and other related characteristics.

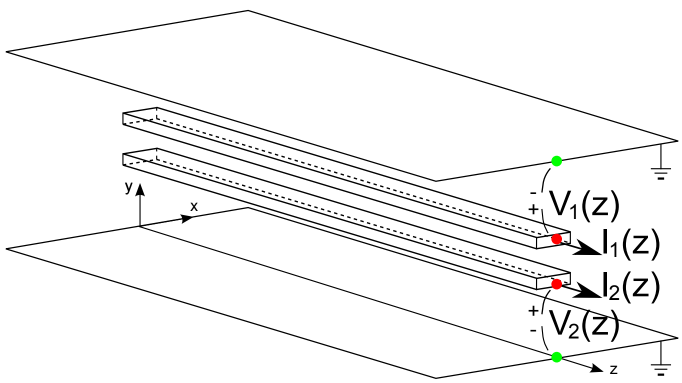

Figure 1. Two broadside coupled dual-striplines.

For a pair of coupled lines, such as two broadside coupled dual stripline traces as shown in 1, by definition, the voltage/current and RLGC parameters satisfy the Telegrapher's equation

$$\begin{aligned} \frac{\partial}{\partial z}\mathbf{V} &= -(\mathbf{R} + j\omega\mathbf{L})\mathbf{I} \\ \frac{\partial}{\partial z}\mathbf{I} &= -(\mathbf{G} + j\omega\mathbf{C})\mathbf{V} \end{aligned} \tag{1}$$

where

$$\begin{aligned} V &= \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T \\ I &= \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T \end{aligned} \tag{2}$$

With the differential- and common-mode variables defined as

$$
\begin{aligned}
v_d &\equiv v_1 - v_2 \\
v_c &= \tfrac{1}{2}(v_1 + v_2) \\
i_d &\equiv \tfrac{1}{2}(i_1 - i_2) \\
i_c &\equiv i_1 + i_2
\end{aligned}
\tag{3}
$$

In matrix form

$$
\begin{bmatrix} v_d \\ v_c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4}
$$

and

$$
\begin{bmatrix} i_d \\ i_c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{5}
$$

Conversely,

$$
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} v_d \\ v_c \end{bmatrix} \tag{6}
$$

and

$$
\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} i_d \\ i_c \end{bmatrix} \tag{7}
$$

The Telegrapher's equation is now

$$
\begin{aligned}
\begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \frac{\partial}{\partial z} \begin{bmatrix} v_d \\ v_c \end{bmatrix} &= - \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} i_d \\ i_c \end{bmatrix} \\
\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} \frac{\partial}{\partial z} \begin{bmatrix} i_d \\ i_c \end{bmatrix} &= - \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} v_d \\ v_c \end{bmatrix}
\end{aligned}
\tag{8}
$$

where

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \mathbf{R} + j\omega\mathbf{L}$$
$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \mathbf{G} + j\omega\mathbf{C} \tag{9}$$

Then,

$$\frac{\partial}{\partial z}\begin{bmatrix} v_d \\ v_c \end{bmatrix} = -\begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} i_d \\ i_c \end{bmatrix}$$
$$\frac{\partial}{\partial z}\begin{bmatrix} i_d \\ i_c \end{bmatrix} = -\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix} v_d \\ v_c \end{bmatrix} \tag{10}$$

or

$$\frac{\partial}{\partial z}\begin{bmatrix} v_d \\ v_c \end{bmatrix} = -\begin{bmatrix} Z_{11} - Z_{12} - Z_{21} + Z_{22} & \frac{1}{2}Z_{11} + \frac{1}{2}Z_{12} - \frac{1}{2}Z_{21} - \frac{1}{2}Z_{22} \\ \frac{1}{2}Z_{11} - \frac{1}{2}Z_{12} + \frac{1}{2}Z_{21} - \frac{1}{2}Z_{22} & \frac{1}{4}Z_{11} + \frac{1}{4}Z_{12} + \frac{1}{4}Z_{21} + \frac{1}{4}Z_{22} \end{bmatrix}\begin{bmatrix} i_d \\ i_c \end{bmatrix}$$
$$\frac{\partial}{\partial z}\begin{bmatrix} i_d \\ i_c \end{bmatrix} = -\begin{bmatrix} \frac{1}{4}Y_{11} - \frac{1}{4}Y_{12} - \frac{1}{4}Y_{21} + \frac{1}{4}Y_{22} & \frac{1}{2}Y_{11} + \frac{1}{2}Y_{12} - \frac{1}{2}Y_{21} - \frac{1}{2}Y_{22} \\ \frac{1}{2}Y_{11} - \frac{1}{2}Y_{12} + \frac{1}{2}Y_{21} - \frac{1}{2}Y_{22} & Y_{11} + Y_{12} + Y_{21} + Y_{22} \end{bmatrix}\begin{bmatrix} v_d \\ v_c \end{bmatrix} \tag{11}$$

In separate modes

$$\frac{\partial}{\partial z}v_d = -\left[(Z_{11} - Z_{12} - Z_{21} + Z_{22})i_d + (\tfrac{1}{2}Z_{11} + \tfrac{1}{2}Z_{12} - \tfrac{1}{2}Z_{21} - \tfrac{1}{2}Z_{22})i_c\right]$$
$$\frac{\partial}{\partial z}i_d = -\left[(\tfrac{1}{4}Y_{11} - \tfrac{1}{4}Y_{12} - \tfrac{1}{4}Y_{21} + \tfrac{1}{4}Y_{22})v_d + (\tfrac{1}{2}Y_{11} + \tfrac{1}{2}Y_{12} - \tfrac{1}{2}Y_{21} - \tfrac{1}{2}Y_{22})v_c\right] \tag{12}$$

and

$$\frac{\partial}{\partial z}v_c = -\left[(\tfrac{1}{2}Z_{11} - \tfrac{1}{2}Z_{12} + \tfrac{1}{2}Z_{21} - \tfrac{1}{2}Z_{22})i_d + (\tfrac{1}{4}Z_{11} + \tfrac{1}{4}Z_{12} + \tfrac{1}{4}Z_{21} + \tfrac{1}{4}Z_{22})i_c\right]$$
$$\frac{\partial}{\partial z}i_c = -\left[(\tfrac{1}{2}Y_{11} - \tfrac{1}{2}Y_{12} + \tfrac{1}{2}Y_{21} - \tfrac{1}{2}Y_{22})v_d + (Y_{11} + Y_{12} + Y_{21} + Y_{22})v_c\right] \tag{13}$$

Ingoring the modal conversion terms caused by the minor asymetry, Eqns.(12) and (13) are reduced to decoupled equations for differential and common modes:

$$\frac{\partial}{\partial z} v_d = -(Z_{11} - Z_{12} - Z_{21} + Z_{22}) i_d$$
$$\frac{\partial}{\partial z} i_d = -\frac{1}{4}(Y_{11} - Y_{12} - Y_{21} + Y_{22}) v_d$$

and

$$\frac{\partial}{\partial z} v_c = -\frac{1}{4}(Z_{11} + Z_{12} + Z_{21} + Z_{22}) i_c$$
$$\frac{\partial}{\partial z} i_c = -(Y_{11} + Y_{12} + Y_{21} + Y_{22}) v_c$$

In other words,

$$\begin{aligned} Z_d &= Z_{11} - Z_{12} - Z_{21} + Z_{22} \\ Y_d &= \tfrac{1}{4}(Y_{11} - Y_{12} - Y_{21} + Y_{22}) \end{aligned} \tag{14}$$

and

$$\begin{aligned} Z_c &= \tfrac{1}{4}(Z_{11} + Z_{12} + Z_{21} + Z_{22}) \\ Y_c &= Y_{11} + Y_{12} + Y_{21} + Y_{22} \end{aligned} \tag{15}$$

2 Differential and Common Mode Characteristic Impedances

Analogous to the relations in a single transmission-line, where the characteristic impedance is assciated with the per-unit-length parameters by

$$Z_{diff,char} = \sqrt{\frac{Z_d}{Y_d}} = 2\sqrt{\frac{Z_{11} - Z_{12} - Z_{21} + Z_{22}}{Y_{11} - Y_{12} - Y_{21} + Y_{22}}}$$

and

$$Z_{comm,char} = \sqrt{\frac{Z_c}{Y_c}} = \frac{1}{2}\sqrt{\frac{Z_{11} + Z_{12} + Z_{21} + Z_{22}}{Y_{11} + Y_{12} + Y_{21} + Y_{22}}}$$

where the $Z_{ij}$and Y_{ij} elements are defined by Eqns.(9). In general, those characteristic impedance values are frequency dependent. For a low-loss system, the characteristic impedance is often used (a constant). For the two separate modes,

$$Z^{\infty}_{diff,char} = \lim_{\omega \to \infty} Z_{diff,char} = 2\sqrt{\frac{L_{11} - L_{12} - L_{21} + L_{22}}{C_{11} - C_{12} - C_{21} + C_{22}}}, \tag{16}$$

and

$$Z^{\infty}_{comm,char} = \lim_{\omega\to\infty} Z_{comm,char} = \frac{1}{2}\sqrt{\frac{L_{11}+L_{12}+L_{21}+L_{22}}{C_{11}+C_{12}+C_{21}+C_{22}}}. \tag{17}$$

Note that the RLGC model parameters from a 2D transmission solver usually have

$$\begin{aligned} &L_{ij} > 0 \\ &C_{ii} > 0, C_{ij} < 0, j \neq i \end{aligned}$$

which is consistent with Eqns.(18) and (19).

2.1 Odd and Even Modes

In RF/microwave engineering field, the term 'odd-mode' and 'even-mode' are used in places of the differential and common modes. The odd- and even-modes are related to the differential- and common-mode by a factor of 2, specifically

$$Z^{\infty}_{odd,char} = \sqrt{\frac{L_{11}-L_{12}-L_{21}+L_{22}}{C_{11}-C_{12}-C_{21}+C_{22}}}, \tag{18}$$

and

$$Z^{\infty}_{even,char} = \sqrt{\frac{L_{11}+L_{12}+L_{21}+L_{22}}{C_{11}+C_{12}+C_{21}+C_{22}}}. \tag{19}$$

2.2 Formulation Under Complete Symmetry

The above derivations are applicable for coupled lines that are not completely symmetric. The results can be further simplified when symmetry is assumed. This is achieved by setting

$$\begin{aligned} &Z_{11} = Z_{22},\ Z_{12} = Z_{21} \\ &Y_{11} = Y_{22},\ Y_{12} = Y_{21} \end{aligned} \tag{20}$$

in the above formulas.